Growth-Collapse Cycles of a Bose-Einstein Condensate with Attractive Interactions

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A Bose-Einstein condensate of atoms with attractive interactions exhibits growth and collapse cycles, when it is fed by a thermal cloud. Recently this phenomenon has been directly observed in a trapped ^{7}Li gas. We offer a quantitative explanation of the data, on the basis of a model proposed earlier. It is shown that the condensate wave function acquires a chaotic component after the first collapse, indicating superfluid turbulence.

I. INTRODUCTION

In an elegant experiment [1], Hulet and his team at Rice University have observed the growth and collapse of a Bose-Einstein condensate of magnetically trapped ^{7}Li atoms, whose interactions are predominantly attractive. This fascinating system had been created [2] and studied [3] by the same group some time ago; but the instability of the system has not been directly observed until now. The purpose of this paper is to give a quantitative explanation of the data, based on a theoretical model proposed earlier [4] [5]. We refer to these references for earlier literature.

Because of the attractive interactions,^{7}Li atoms in free space would solidify at low temperatures. When confined in an external potential, however, the atoms are maintained in a gaseous state by their zero-point motion, as long as their number is small. Above a critical number, however, the attractive interactions overcome the kinetic energy, and the system collapses into a state of high density. This instability is similar to that of a white dwarf star whose mass is greater than the Chandrasekhar limit, in which the gravitational attraction overwhelms the zero-point pressure of the Fermi gas of electrons in the star. Unlike in a white dwarf star, however, the particle number here is not conserved, for the atoms can be expelled from the magnetic trap due to two- and three-body collisions, which become increasingly important at higher densities. During the collapse, the particle number suddenly drops to a subcritical value, only to grow back subsequently, if the condensate is fed by a surrounding thermal cloud. Thus, the condensate undergoes cycles of growth and collapse, and in this respect the system is more akin to an electric buzzer than a white dwarf star.

Let us first ignore the gain and loss mechanisms. To model the system, we treat the condensate in the mean-field approximation, an assume that the condensate wave function ψ satisfies a Gross-Pitaevskii equation, or nonlinear Schrödinger equation (NLSE):

$$
\begin{aligned}
i\hbar\frac{\partial\psi}{\partial t} &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) - U_0|\psi|^2\right]\psi, \\
U_0 &= \frac{4\pi\hbar^2|a|}{m},
\end{aligned} \tag{1}
$$

where a is a negative scattering length, and the external potential is taken to be harmonic:

$$V(r) = \frac{1}{2}m\omega^2 r^2. \tag{2}$$

This defines a characteristic length d_0, the width of the unperturbed ground-state wave function:

$$d_0 = \sqrt{\frac{\hbar}{m\omega}}. \tag{3}$$

The number of condensate particles enters through the normalization

$$N = \int d^3r|\psi|^2. \tag{4}$$

The corresponding Hamiltonian is given by

$$H = \int d^3r \left[-\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V(r)\psi^*\psi - \frac{U_0}{2}(\psi^*\psi)^2 \right]. \tag{5}$$

In the Rice experiment, the parameters have the values

$$\begin{aligned} a &= -1.45 \text{ nm}, \\ d_0 &\approx 3.16\ \mu\text{m}, \\ \omega &\approx 908 \text{ s}^{-1}. \end{aligned} \tag{6}$$

Actually, the magnetic trap was not quite spherically symmetric, and ω here represents the geometric mean of the frequencies.

In the absence of an external potential in spatial dimensions $D \geq 2$, the NLSE with attractive interactions ($U_0 > 0$) exhibits "self-focusing" [6], in which the system collapses to a state of locally infinite density in a finite time. The cause of this instability is the term $-\frac{1}{2}U_0|\psi|^2$ in the Hamiltonian 5, which has no lower bound as the density $|\psi|^2$ increases. In the presence of an external trap $V(r)$, however, there exists an energy barrier against collapse. As a function of $|\psi|^2$, the form of this barrier depends on the location r, and its height is smallest at $r = 0$, where the external potential is weakest. As predicted in [4], and verified through numerical calculations in [5], the energy barrier at $r = 0$ is breached when $N > N_c$, with

$$N_c = 0.574\frac{d_0}{|a|} = 1250, \tag{7}$$

and this triggers the collapse of the system. At the center of the trap, there appears a black hole into which particles are drawn, and they will be drained from the system when loss mechanisms are taken into account. Of course, particles can surmount the energy barrier through quantum tunneling; even when $N < N_c$. but this is a slow process that can be neglected compared to loss through collisions.

What happens to the particles that go through the energy barrier? According to the Hamiltonian 5 their density would continue to increase indefinitely; but, of course, once the density begins to grow, effects so far neglected will become important. An example is solidification, which might be simulated by adding high-order terms in the Hamiltonian, such as $c|\psi|^6$, with $c > 0$. Such a term in the NLSE will lead to the formation of droplets in a first-order phase transition [7]. However, such effects are overshadowed by loss though collisions, and will be ignored.

II. GAIN AND LOSS MECHANISMS

Ketterle and his team at MIT has measured the rate of growth of a ^{23}Na condensate fed by a thermal cloud, and fitted the results to the following formula [8]:

$$\frac{dN}{dt} = c_0 N \left[1 - \left(\frac{N}{N_{\text{eq}}} \right)^{0.4} \right], \tag{8}$$

with $c_0 = (50 \text{ ms})^{-1}, N_{\text{eq}} \approx 5 \times 10^5$. The factor in square brackes represents a saturation effect. We shall ignore it by assuming $N << N_{\text{eq}}$ for our case. The growth rate c_0 should be proportional to the elastic scattering cross section, and hence to the square of the scattering length. Since the scattering lengths for ^{23}Na and ^{7}Li are 2.75 nm and -1.45 nm respectively, we take for ^{7}Li

$$c_0 = \left(\frac{1.45}{2.75} \right)^2 (50 \text{ ms})^{-1}. \tag{9}$$

The loss rate $R(N)$ due to two and three-body collisions have been calculated by Dodd *et al [9],* who report their results in the form

$$\begin{aligned} R(N) &= A \int d^3r|\psi|^4 + B \int d^3r|\psi|^6, \\ A &= 1.2 \times 10^{-14} \text{ cm}^3 \text{ s}^{-1}, \\ B &= 2.6 \times 10^{-28} \text{ cm}^6 \text{ s}^{-1}, \end{aligned} \tag{10}$$

where A is due to two-body collisions, and B is due to three-body collisions. The experimental value of the two-body loss rate A is consistent with the value give above, provided it is interpreted as the loss rate per event [10]. In view of this fact, we reinterpret both A and B as rates *per event* rather than per particle. Since 2 particles are lost in a two-body collision, and 3 particles are lost in a three-body collision, the loss rates per particle is taken to be $2A$ for two-body collisions, and $3B$ for three-particle collisions [11].

The gain and loss rates can be taken into account by adding absorptive terms to the NLSE:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) - U_0|\psi|^2\right]\psi + \frac{i\hbar}{2}\left(\gamma - 2A|\psi|^2 - 3B|\psi|^4\right)\psi. \tag{11}$$

The initial wave function is taken to be the ground-state eigenfunction in the harmonic potential:

$$\psi_0(r) = \pi^{-3/4}d_0^{-3/2}\sqrt{N_0}\exp\left(-r^2/d_0^2\right), \tag{12}$$

where N_0 is the initial number of particles. Note that the coefficients A and B were defined as rates rather than coupling constants, and do not need corrections factors arising from Bose symmetry.

We assume that ψ is spherically symmetric, and put

$$\begin{aligned}\tau &= \frac{1}{2}\omega t,\\ \rho &= \frac{r}{d_0},\\ \psi &= \sqrt{\frac{N}{4\pi}}d_0^{-3/2}\frac{u}{\rho}.\end{aligned} \tag{13}$$

The dimensionless NLSE for the reduced wave function $u(\rho,\tau)$ reads

$$\frac{\partial u}{\partial\tau} = i\left[\frac{\partial^2}{\partial\rho^2} - \rho^2 + 2g\frac{|u|^2}{\rho^2}\right]u + \left[\gamma - g\alpha\frac{|u|^2}{\rho^2} - g^2\beta\frac{|u|^4}{\rho^4}\right]u, \tag{14}$$

where

$$\begin{aligned}g &= \frac{|a|}{d_0} = 4.87\times10^{-4},\\ \gamma &= \frac{\gamma}{\omega} = 6.18\times10^{-3},\\ \alpha &= \frac{A}{2\pi\omega d_0|a|} = 1.46\times10^{-4},\\ \beta &= \frac{3B}{(4\pi)^2\omega d_0^4|a|^2} = 2.616\times10^{-5}.\end{aligned} \tag{15}$$

The instantaneous particle number, given by

$$N(\tau) = \int_0^\infty d\rho|u(\rho,\tau)|^2, \tag{16}$$

is no longer a constant of the motion. The initial wave function is chosen to be

$$u(\rho,0) = 2\pi^{-1/4}\sqrt{N_0}\rho\exp\left(-\rho^2/2\right). \tag{17}$$

The initial particle number $N_0 = N(0)$ is the only adjustable parameter of the model.

III. COMPARISON WITH EXPERIMENTS

We solve the modified NLSE (14) numerically, using an algorithm due to Goldberg *et al* [12], which is a version of the popular Crank-Nicholson method [13]. The computational spatial lattice has size $L\rho = 10000$, with spacing

$d\rho = 0.0008$, which corresponds to 2.53×10^{-7} cm. The temporal grid is of size $L_\tau = 40000$, with spacing $d\tau = 0.017$, which corresponds to 3.74×10^{-5} s. The boundary conditions are $u(0, \tau) = u(L_\rho, \tau) = 0$.

The Rice experiment [2] consists of three groups of measurements of $N(\tau)$ with different N_0. The initial values were quoted as $N_0 = 500, 100, 0$, with errors of ± 60. Each data point was the average of 5 separate runs. We fit the data by choosing the best N_0 for each group, and also perform an average over a uniform distribution of N_0 centered about the chosen value, with a width adjusted to yield the best visual fit. The three groups of calculations correspond to the following distributions:

Group	$\langle N_0 \rangle$	Distribution
I	325	$385, 355, 325, 295, 265$
II	75	$85, 80, 75, 70.65$
III	31	$61, 46, 31.16, 1$

The comparison with data is shown in Fig.1. For different N_0, the calculated $N(\tau)$ has practically the the same functional form, except that the time origin is shifted. Typically, $N(\tau)$ rises exponentially to a maximum of 1200, and then suddenly collapses to about 450. Thereafter, it exhibits a periodic saw-tooth pattern of growth and collapse. When averaged over a uniform distribution of N_0, the initial rise remains unchanged, but the subsequent cycles tend to be smoothed out. The experimental distribution was surely not uniform, but no attempts were made to vary the distribution function to achieve a more precise fit.

The behavior of the initial rise depends mainly on the growth coefficient γ, and is insensitive to the loss coefficients α and β. No discernible difference is found by varying α and β by factors of 2 to 3. On the other hand, the behavior after the first collapse is sensitive to α, β, and to variations of the initial wave function. Unfortunately, the data in that region is not good enough for comparison with theory.

In our model, we have neglected quantum tunneling, the saturation of the gain rate, trap asymmetry, and other effects. The fact that we can fit the data indicates that these effects are of secondary importance.

IV. DETAILS OF THE COLLAPSE PROCESS

We can understand some features of the collapse process by examining the wave function just before and after the collapse. In order to mark the time more precisely, we show in Fig.2 a plot of N as a function of time t, which is measured in units of $50\, d\tau = 1.87 \times 10^{-3}$ s. With an initial value $N_0 = 325$, the first collapse happens at $t = 127$.

In Fig.3, we show the reduced wave functions $u(x, t)$ for $t = 100, 127, 128$. The position x is the site number x of the computational lattice. We see that, as time goes on, the peak of the wave function narrows as it migrates to smaller x, reaching $x = 300$ at $t = 127$, when the collapse begins. At $t = 128$ the peak has moved to $x = 5$, as shown in the inset in Fig.3. The particles under the peak are thus suddenly squeezed into a state of very high density. This is in the region referred to as the black hole. The area under the peak, minus background, corresponds to about 750 partilces. The peak has disapeared by $t = 130$, as we can see in Fig.4. This indicates that the loss mechanisms, which are proportional to high powers of the density, became activated suddenly, and drained the black hole. Once this happens, the density drops, and the loss mechanisms become dormant. The particle number grows due to the gain mechanicm, and the wave function slowly recovers an approximately Gaussian shape, as seen in Fig.4.

According to Fig.1, $1200 - 450 = 750$ particles were lost during the first collapse. This is consistent with loss in the balck hole, and indicates that loss mechanisms are effective only in the black hole dring a very brief time interval. This explains why the behavior of $N(t)$ is not sensitive to the loss coefficients. In our coarse-grained time, the interval of loss amount to 0.004 s. Although this is very short compared to the oscillator period, it is long compared to atomic collison times. A study at smaller time scales would require going beyond the mean-field approxiamtion. Work in this direction has been initiated by Saito and Ueda [14].

As we can see from Fig.4, after the first collapse, the wave function acquires a persistent chaotic component. This means that the wave function it is very sensitive to variations in the parameters of the model, and the initial wave function, as has been pointed out [5]. In Fig.5, we examine the chaotic feature moser closely by plotting at different times the superfluid velocity field

$$v_s = \frac{\hbar}{m} \frac{\partial \varphi}{\partial r}, \tag{18}$$

where φ is the phase of the condensate wave function. At the onset of collapse at $t = 127$, the velocity is everywhere negative, indicating a flow towards the center. It becomes oscillatory at $t = 128$, and present a more an more chaotic look as time goes on. This suggests that the ^{7}Li condensate may be a good medium for experimental and theoretical studies of superfluid turbulence.

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Figure Captions

Fig.1. Grow-collapse cycles of the particle number. Data points are solid circles, with error bars omitted for clarity. The statistical error is typically 20%. Thick solid lines are predictions of the model for given N_0. The dotted lines indicate results of averaging over a uniform distribution of N_0 described in the text.

Fig.2. Particle number as function of. time t measure in units of 50 computational steps, corresponding to 1.87×10^{-3} s.

Fig.3. The reduced condensate wave function just before and just after the collapse. The position x is the site number on the computational grid, with a spacing corresponding to 2.53×10^{-7} cm. The inset shows behaviors near the origin, in the black hole. At $t = 128$, there are about 750 particles under the peak at $x = 5$. They are purged from the system during the next time step, in about 0.002 s.

Fig.4. After the collapse, the loss mechanism become inactive, and the system gains particles from the thermal cloud. The wave function recovers an approximately Gaussian shape, but a chaotic component remains.

Fig.5. The superfluid velocity field at various times. There is a flow towards the center just before the collapse at $t = 127$. Thereafter it looks increasing turbulent.

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